Physics 525, Condensed Matter

Homework 8

Due Thursday, 14^{th} December 2006

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Problem 1: Little-Parks Experiment

Consider a long, thin-walled, hollow cylinder of radius R and thickness d made of a superconductor subjected to an external magnetic field H which is parallel to the axis of the cylinder. If the wave function for superconducting electron pairs $\Psi(r)$ is taken as the order parameter for a Landau-Ginzburg theory, the free energy density is then

$$f = f_n + a(T - T_{c_0})|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{\hbar^2}{2m}\left|\left(-i\vec{\nabla} + \frac{2e}{\hbar c}\vec{A}\right)\Psi\right|^2 + \frac{1}{8\pi}\left(\vec{H} - \vec{B}\right)^2. \tag{1.1}$$

Here, we have made the additional assumption that the leading coefficient is a linear function in T that vanishes at the zero-field critical temperature T_{c_0}. To lowest order in the induced current, $\vec{H} = \vec{B}$ everywhere, and the last term of the free energy may be ignored.

a. We are to find the lowest free-energy state as a function of both T and H, for H such that the total flux through the interior of the cylinder is no more than a few flux quanta. We are to give the values of $|\Psi(r)|$ and $\vec{v}_s(r)$ and calculate the shifted critical temperature as a function of applied field.

Let the cylinder lie with its axis in the $\hat{z}$-direction in cylindrical coordinates. We may write choose our gauge so that the vector potential is

$$\vec{A} = \frac{rH}{2}(0, 1, 0), \tag{1.2}$$

which is easily seen to give rise to $\vec{H} = H\hat{z}$.

Before we try to find the minimum of the free energy, it will be helpful to cut away generality of our analysis for the conveniences offered by the case at hand. Recall that the field within a thin-walled superconductor is well approximated by London theory; this is because gradient terms in the magnitude $|\Psi|$ cost too much free energy when the field must vanish outside of the thin cylinder. Therefore, we may write

$$\Psi(r) = \psi e^{i\varphi(r)}, \qquad \text{for} \quad \psi \in \mathbb{R}. \tag{1.3}$$

Of course, Ψ must be taken to vanish outside the superconductor, but this will not really complicate our analysis. Observe that

$$(-i\vec{\nabla}\Psi) = \Psi\left(\vec{\nabla}\varphi(r)\right), \tag{1.4}$$

which encourages us to write the fourth term in (1.1) as

$$\frac{\hbar^2}{2m}\left|\left(-i\vec{\nabla} + \frac{2e}{\hbar c}\vec{A}\right)\Psi\right|^2 = \frac{\hbar^2}{2m}|\Psi|^2\left(\vec{\nabla}\varphi(r) + \frac{2e}{\hbar c}\vec{A}\right) \equiv \frac{1}{2}m\psi^2\vec{v}_s^2, \quad \text{where} \quad \vec{v}_s \equiv \frac{\hbar}{m}\left(\vec{\nabla}\varphi(r) + \frac{2e}{\hbar c}\vec{A}\right). \tag{1.5}$$

Using the symmetry of the problem it is obvious that there are only currents in the $\hat{\theta}$-direction; this implies that $\partial_r(\varphi) = \partial_z(\varphi) = 0$[1]; so φ is only a function of θ. Furthermore, single-valuedness of the wave function requires that

$$\oint \nabla\varphi = 2\pi r\frac{1}{r}\partial_\theta\varphi = 2\pi n, \quad \Longrightarrow \quad \partial_\theta\varphi = n, \quad \Longrightarrow \quad \varphi(\theta) = n\theta. \tag{1.6}$$

Using this together with the definition

$$\Phi_0 \equiv \frac{hc}{2e}, \tag{1.7}$$

and the fact that the flux through the cylinder is $\Phi = H\pi R^2$, we can write

$$v_s = \frac{\hbar}{mR}\left(n + \frac{\Phi}{\Phi_0}\right). \tag{1.8}$$

[1] Actually, symmetry does not exclude currents in the $\hat{z}$-direction: these are excluded because they raise the free energy unnecessarily—so that the ground state will have no vertical components.

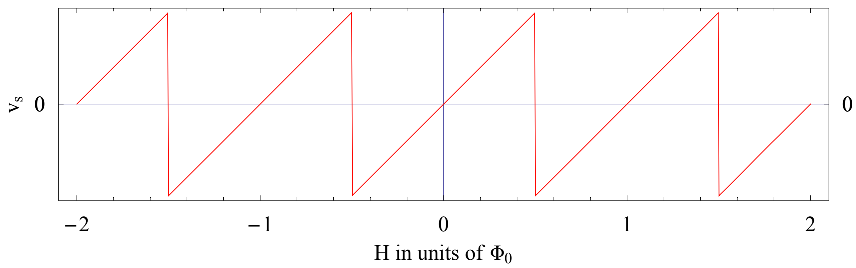

FIGURE 1. The velocity operator v_s as a function of the applied field H.

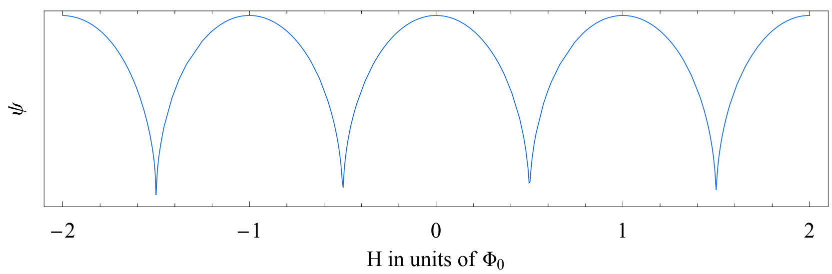

FIGURE 2. The magnitude of the wave funciton $\psi(H)$.

Putting everything together, we find the free energy of the superconductor to be

$$f_s = a(T - T_{c_0})\psi^2 + \frac{\beta}{2}\psi^4 + \frac{1}{2}mv_s^2\psi^2. \tag{1.9}$$

Notice that the free energy is naturally lowered by seeking the smallest possible value of v_s. This is done by choosing the integer n so that $n - \frac{\Phi}{\Phi_0}$ is minimized. This is shown in Figure 1. Minimizing this with respect to the field magnitude ψ, we obtain the Landau-Ginzburg equation

$$\left\{a(T - T_{c_0}) + \beta\psi^2 + \frac{1}{2}mv_s^2\right\}\psi = 0. \tag{1.10}$$

$$\therefore \psi^2 = \frac{1}{\beta}\left(a(T_{c_0} - T) - \frac{1}{2}mv_s^2\right). \tag{1.11}$$

The parametric dependence on H is shown in Figure 2.

If we define $T_c(H)$ to be the field temperature at which ψ^2 vanishes for a given H (which enters our expression via v_s), we see that

$$T_c(H) = T_{c_0} - \frac{mv_s^2}{2a}. \tag{1.12}$$

This was one of the principle experimental results of Little and Parks. The dependence of T_c on H is shown in Figure 3.

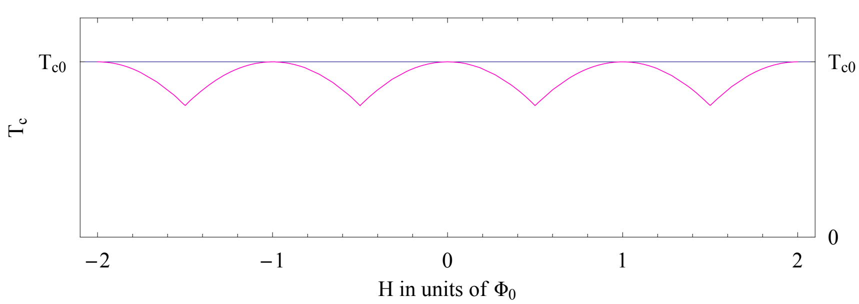

FIGURE 3. The critical temperature as a function of H.

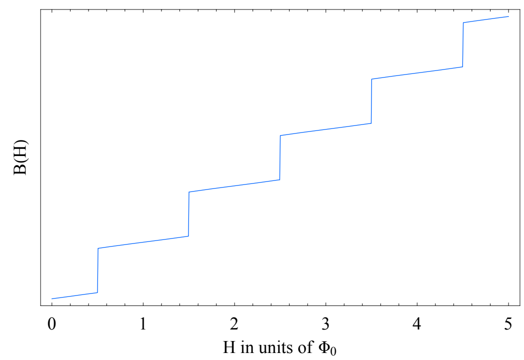

FIGURE 4. The qualitative structure of $B(H)$.

b. We are to obtain the field $\vec{B}(H)$ to leading order in the current of the cylinder and describe when this approximation is valid.

Recall that the current density is

$$J = -2e\psi^2 v_s = -\frac{2e}{\beta} v_s \left(a(T_{c_0} - T) - \frac{mv_s^2}{2} \right). \tag{1.13}$$

We can find the (upper bound on) the maximum of J by differentiating with respect to v_s; we find

$$v_s^2 = \frac{2a(T_{c_0} - T)}{3m}, \tag{1.14}$$

will maximize J. The reason why one could be worried is that v_s is of course bounded. However, the bound on v_s extends beyond the v_s required to saturate the maximum of J,

$$v_s^2 \leq \frac{2a(T_{c_0} - T)}{m}. \tag{1.15}$$

Anyway, we may use this to find the maximum current,

$$J_{max} = -4e\frac{a(T_{c_0} - T)}{\beta}\sqrt{\frac{2a(T_{c_0} - T)}{3m}}. \tag{1.16}$$

Using the equation above for the current as a function of v_s, we can compute the induced B-field as a function of H. The total field is shown in Figure 4.

Problem 2: Plane Waves on Thin Superconductors

Consider a thin-sheet superconductor with the same Landau-Ginzburg free energy expansion as discussed in problem 1. We are to determine the spatially uniform configurations of $\Psi(r)$ which are minima of the free energy under variation of ψ. We should determine the maximum current that can be carried by the wave—and especially describe the behaviour as T_{c_0} is approached from below.

Similar to the situation above, we notice that solutions will necessarily have no spatial variation in magnitude—again, because this gradient term costs too much in the free energy. Therefore, we may write (approximate) any solution which minimizes the Landau Ginzburg free energy as $\Psi(r) = \psi e^{i\varphi(r)}$ for $\psi \in \mathbb{R}$. And also like above we find that—ignoring the magnetic fields induced by the supercurrent—

$$\psi^2 = \frac{1}{\beta}\left(a(T_{c_0} - T) - \frac{1}{2}mv_s^2 \right), \tag{2.1}$$

where

$$\vec{v}_s(r) \equiv \frac{\hbar}{m}\vec{\nabla}(\varphi(r)). \tag{2.2}$$

Notice that this time there is no contribution from the vector potential.

A spatially uniform solution must therefore be one such that $\vec{v}_s(r) = k \cdot x$ for some constant vector k inside the thin-sheet superconductor and $\hat{x}$ is a direction in the plane—which we may take as one of the coordinate axes for convenience. Such a solution will manifestly generate a spatially uniform velocity,

$$\vec{v}_s = \frac{\hbar}{m} k\hat{x}. \tag{2.3}$$

Recalling some standard notation,

$$\psi_0^2 = \frac{a(T_{c_0} - T)}{\beta}, \qquad \text{and} \qquad \xi^2(T) = \frac{\hbar^2}{2ma(T_{c_0} - T)}, \tag{2.4}$$

we can rewrite the minimization condition as

$$\psi^2 = \psi_0^2 \left(1 - \frac{\hbar^2 k^2}{2ma(T_{c_0} - T)}\right) = \psi_0^2 \left(1 - \xi^2 k^2\right). \tag{2.5}$$

Therefore, we see that spatially uniform solutions are of the form

$$\Psi(r) = \psi_0 \sqrt{1 - \xi^2 k^2} e^{ikx}. \tag{2.6}$$

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Notice that this requires that

$$k < \frac{1}{\xi} \tag{2.7}$$

for a superconducting solution.

This gives us a current of

$$J = -2e\psi^2 v_s = -2e\psi_0^2 k \frac{\hbar}{m} \left(1 - \xi^2 k^2\right). \tag{2.8}$$

The extreme current can be found via differentiation

$$1 - 3\xi^2 k^2 = 0 \qquad \Longrightarrow \qquad k = \frac{1}{\sqrt{3}\xi}; \tag{2.9}$$

which of course implies that

$$\therefore J_{max} = \psi_0^2 \frac{1}{\xi} \frac{2\hbar}{3\sqrt{3}m}. \tag{2.10}$$

Near $T = T_{c_0}$ from below, we know that

$$\psi_0 \sim 1 - \frac{T}{T_{c_0}}, \qquad \text{and} \qquad \xi(T) \sim \frac{1}{\sqrt{1 - T/T_{c_0}}}, \tag{2.11}$$

and therefore the maximum current vanishes according to

$$J_{max} \sim \left(1 - \frac{T}{T_{c_0}}\right)^{3/2}. \tag{2.12}$$

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Problem 3: Superconducting Spheres and Monopoles

Aluminium is a type-I superconductor with coherence length $\xi = 1.5 \ \times 10^{-6}$m and a magnetic penetration length $\lambda = 2 \times 10^{-8}$m at low temperature.

a. At zero temperature, a spherical piece of aluminium of radius $r \gg \lambda$ is placed in a magnetic field $\vec{B} = B(z)\hat{z}$ with a small field gradient ($r \gg dB/dz$) and $B < H_c$. We are to state $H_c(T = 0)$ and compute the force on the sphere for the specific case $r = 10^{-3}$ m, $B = 10^{-3}$ T, and $dB/dz = 10^{-2}$ T/m and compare this to the case if $B \mapsto 1$ T while keeping everything else the same.

First, we notice that

$$H_c = \frac{\Phi_0}{2\sqrt{2}\pi\xi(0)\lambda(0)} = 7.8 \times 10^{-3} \text{ T} = 78 \text{ gauss}. \tag{3.1}$$

To compute the force of the sphere of superconducting aluminium from the B-field, we need to find the external field. This is done by first finding the field inside the superconductor. There, we know that $\vec{B}$ must vanish. Because gradient is small compared to the radius of the sphere, we can approximate the solution by considering the field to be constant throughout the sphere. It is clear that the superconducting currents on the surface must cancel this field exactly inside. Therefore, we can quickly discover the required field arrangement if we know of a system which gives rise to a uniform magnetic inductance within the volume of a sphere.

Now, if we remember our electrodynamics as much as any student studying for Preliminary exams should, then we recall at once that such a field distribution is obtained by a rotating, insulating sphere of uniform surface charge density. We could derive this solution but, for the sake of brevity, we will refer the reader to homework solutions prepared many years ago by the author[2]. Alternatively, we draw the reader's attention to section (5.10) of Jackson's *Classical Electrodynamics*.

The principle result was that if a sphere has a uniform magnetic field in its interior given by

$$B\hat{z} = \frac{2\mu_0}{3}M\hat{z}, \tag{3.2}$$

then the field outside the sphere is precisely that of a dipole with dipole moment

$$m\hat{z} = \frac{4\pi r^3}{3}M\hat{z}. \tag{3.3}$$

In our present situation, we need the field within the sphere to exactly cancel that of the ambient B-field. This tells us that currents which generate a uniform $B_{in} = -B(z)$ give rise to a perfect dipole field outside the sphere with dipole moment

$$m\hat{z} = \frac{4\pi r^3}{3}\frac{3}{2\mu_0}(-B(z)) = -\frac{2\pi r^3}{\mu_0}B(z). \tag{3.4}$$

Making use again of the fact that the sphere is small compared to the gradient of B, we will not lose much by considering the interaction between B and the sphere therefore to be that between a any magnetic inductance and a dipole—namely,

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = -\frac{2\pi r^3}{\mu_0}B(z)\frac{\partial B(z)}{\partial z}. \tag{3.5}$$

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Putting in the numbers for our problem, we find

$$F(10^{-3}\ \mathrm{T}) = 10^{-2}\ \text{dyne}, \qquad \text{and} \qquad F(1\ T) = 10\ \text{dyne}. \tag{3.6}$$

b. Consider a Dirac monopole/anti-monopole pair 10^{-2} m apart within a much larger piece of superconducting aluminium at zero temperature. The monopole emits magnetic flux hc/e. We are to give a rough estimate of the force between these two hypothetical particles.

The physical picture to have in mind is the following. If the two monopoles were separated in free-space, their flux lines would be exactly analogous electrostatics. However, magnetic flux by definition cannot exist within a superconducting state—the superconductor will do all that it can to confine the magnetic flux to a very small region in the superconductor. Therefore, it is easy to imagine that *all* of the flux connecting the two monopoles is *confined* to a narrow 'sting' between the two monopoles. The width of this string is roughly 2λ—because this is as narrow a region as a Type-I superconductor can confine a region of non-critical state.

We may approximate the situation as there being a cylindrical band connecting the two monopoles in which there is confined all of the magnetic flux between them and completely normal-state Aluminium. In terms of energy costs, the flux lines must

[2] If the grader truly desires to see this calculation, please see the homework prepared during a course in the fall of 2004 at: `http://www.umich.edu/~jbourj/jackson/5-13.pdf`.

pay a debt of free energy of[3]

$$F_{mag} = \frac{1}{8\pi} d\pi\lambda^2 \left(\frac{2hc}{e\pi\lambda^2}\right)^2 = \frac{2d\Phi_0^2}{\pi^2\lambda^2}, \tag{3.7}$$

where have made use of the flux quantum Φ_0. Also, we must pay the energy debt of raising the superconducting minimum to the normal state over the volume of the 'flux tube' of force. Using some identities from work elsewhere, we have as the leading term in the Landau Ginzburg free-energy potential,

$$F_s = d\pi\lambda^2\alpha\psi_0^2 = d\pi\xi^2 \frac{2e^2 H_c^2(T)\lambda^2}{mc^2} \frac{mc^2}{8\pi e^2\lambda^2} = d\frac{\xi^2}{4} H_c^2. \tag{3.8}$$

Notice that this energy is linear with distance: this is as expected: the energy content of the flux tube is proportional to its length. We find the force then to be

$$F = -\left(\frac{2\Phi_0^2}{\pi^2\lambda^2} + \frac{\xi^2}{4} H_c^2\right). \tag{3.9}$$

This is an attractive force on the order of a dyne.

[3]I'm not sure exactly the choice of units here, but if I compte Φ_0 in Tesla-meters^2, then there needs to be a $1/\mu_0$ to get the units right. I suspect that $\frac{1}{8\pi} \mapsto 1/\mu_0$, but it doesn't really matter: we are only asked to qualitatively describe the force.